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# TEICHMÜLLER MAPPING CLASS GROUP OF THE UNIVERSAL HYPERBOLIC SOLENOID

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Abstract. We show that the homotopy class of a quasiconformal self-map of the universal hyperbolic solenoid $H_\infty$ is the same as its isotopy class and that the uniform convergence of quasiconformal self-maps of $H_\infty$ to the identity forces them to be homotopic to conformal maps. We identify a dense subset of $\mathcal{T}(H_\infty)$ such that the orbit under the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ of any point in this subset has accumulation points in the Teichmüller space $\mathcal{T}(H_\infty)$. Moreover, we show that finite subgroups of $MCG_{BLP}(H_\infty)$ are necessarily cyclic and that each point of $\mathcal{T}(H_\infty)$ has an infinite isotropy subgroup in $MCG_{BLP}(H_\infty)$.

## 1. Introduction

The universal hyperbolic solenoid $H_\infty$ was introduced by Sullivan in [13]. Its definition is motivated by dynamics. The solenoid $H_\infty$ is a fiber space over a closed Riemann surface of genus greater than 1 which is locally homeomorphic to a disk times a Cantor set. The path components of $H_\infty$ are called *leaves*. Each leaf is homeomorphic to the unit disk. There is a distinguished leaf, called the *base leaf*. The solenoid $H_\infty$ supports complex structures and the Teichmüller space $\mathcal{T}(H_\infty)$ of $H_\infty$ is a certain closure of the stack of the Teichmüller spaces of all closed surfaces of genus greater than 1 (see [13] and [10]). Also, $\mathcal{T}(H_\infty)$ can be seen as a complex submanifold of the universal Teichmüller space in the sense of Ahlfors-Bers [1]. As usual in Teichmüller theory every point of $T(H_\infty)$ can be represented by a quasiconformal map $f$ and by a complex solenoid $X$, where $f : H_\infty \to X$. In this paper we assume that quasiconformal mappings are smooth. This technical requirement is necessary in order to make a proper definition of continuity for the transversal variation on local charts [13].

It is classical that two homotopic quasiconformal homeomorphisms of a closed Riemann surface are in fact isotopic (this is true in the case of arbitrary non-compact Riemann surfaces; see [6]). Our first result shows that the homotopy class of a quasiconformal self-map of $H_\infty$ is the same as the isotopy class.

**Theorem 3.1.** *Let $f : X \to Y$ and $g : X \to Y$ be two homotopic quasiconformal maps of complex solenoids $X$ and $Y$. Then $f$ and $g$ are isotopic by a uniformly quasiconformal isotopy.* □

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We also show that any homeomorphism $h : H_\infty \to H_\infty$ is homotopic to a quasiconformal self-map of $H_\infty$ (see Theorem 3.2). This property is shared by closed surfaces but is not true for geometrically infinite surfaces, e.g. unit disk.

If a sequence of quasiconformal self-maps of a closed Riemann surface converges uniformly on compact susets to the identity, then all but finitely many of these quasiconformal mappings must be isotopic to the identity. However, most non-compact Riemann surfaces do not have this property. We say that $H_\infty$ is a *transversely locally constant (TLC) complex solenoid* if the complex structure on $H_\infty$ is the lift of the complex structure on a closed Riemann surface under the fiber map. We show that $H_\infty$ satisfies the following.

**Theorem 4.1.** *Let $H_\infty$ be a TLC complex solenoid and let $f_n : H_\infty \to H_\infty$ be a sequence of base leaf preserving quasiconformal self-maps of $H_\infty$ that uniformly converges to the identity map. Then there exists $n_0$ such that $f_n$ is homotopic to a base leaf preserving conformal self-map $c_n : H_\infty \to H_\infty$, for all $n > n_0$.*

The *mapping class group $MCG(H_\infty)$* of the universal hyperbolic solenoid $H_\infty$ consists of isotopy classes of quasiconformal self-maps $h : H_\infty \to H_\infty$. An element $h \in MCG(H_\infty)$ defines a geometric isomorphism $\rho_h : \mathcal{T}(H_\infty) \to \mathcal{T}(H_\infty)$ by $\rho_h([f]) = [f \circ h^{-1}]$, for $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$. The *base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$* consists of all $h \in MCG(H_\infty)$ which preserve the base leaf of $H_\infty$. The virtual automorphism group $Vaut(\pi_1(S))$ of the fundamental group of a closed surface $S$ of genus greater than 1 is isomorphic to $MCG_{BLP}(H_\infty)$ (see [11] and [3]).

**Theorem 5.1.** *There exists a dense subset of $\mathcal{T}(H_\infty)$ such that the orbit of the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ of any point in this subset has accumulation points in $\mathcal{T}(H_\infty)$.*

The mapping class group of a closed surface $S$ acts properly discontinuously on the Teichmüller space $\mathcal{T}(S)$. This is not true in the case of the solenoid:

**Corollary 5.1.** *The base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ does not act properly discontinuously on the Teichmüller space $\mathcal{T}(H_\infty)$.*

Any finite subgroup of the mapping class group of a closed surface $S$ of genus greater than 1 is realized as the isotropy group of a point in the Teichmüller space $\mathcal{T}(S)$ of $S$ [8]. A self-map of a closed surface of genus greater than 1 lifts to a self-map of the solenoid $H_\infty$ (there are infinitely many lifts of a single self-map). The lifted self-map of $H_\infty$ gives an element of $MCG_{BLP}(H_\infty)$, called *mapping class like*. We show, in Theorem 6.1, that any finite subgroup $F$ of $MCG_{BLP}(H_\infty)$ is realized as a subgroup of an isotropy group of a point $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$, where the elements of $F$ are mapping class-like and $X$ is a TLC complex solenoid. In addition, a finite subgroup of $MCG_{BLP}(H_\infty)$ is necessarily cyclic (see Corollary 6.1). Note a direct corollary to Theorem 6.1:

**Corollary 6.2.** *Elements of $MCG_{BLP}(H_\infty)$ which are not mapping class-like are of infinite order.*

Let $X$ be a TLC complex solenoid with its complex structure lifted from a closed Riemann surface $S$. The isotropy subgroup of $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ is isomorphic to the commensurator of a Fuchsian covering group of $S$ in the group of automorphisms of the unit disk (see [11] and [3]).

We consider the isotropy group of an arbitrary $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$, where $X$ is not a TLC solenoid. The holomorphic universal covering of $X$ is the unit disk times a Cantor set [12]. The group of automorphisms of the universal covering space of $X$ consists of all homeomorphisms which are isometries on disks and are continuous for the transversal variation. The universal covering group of $X$ is a subgroup of the group of automorphisms.

**Theorem 7.1.** *Let $f : H_\infty \to X$ be a quasiconformal map, where $X$ is an arbitrary solenoid. Then the isotropy subgroup of $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ is isomorphic to the commensurator group of the covering group of $X$ with respect to the group of automorphisms of the universal covering of $X$.*

The covering group of the solenoid $X$ is a subgroup of its commensurator. Consequently, each $[f : H_\infty \to X \in \mathcal{T}(X)]$ has an infinite isotropy subgroup in $MCG_{BLP}(H_\infty)$.

## 2. The definition and the universal coverings of the solenoid $H_\infty$

We define the universal hyperbolic solenoid $H_\infty$. Fix a pointed closed surface $(S, x)$ of genus greater than 1. Consider all pointed, finite sheeted, unbranched coverings $\pi_i : (S_i, x_i) \to (S, x)$ by closed surfaces $(S_i, x_i)$ such that $\pi_i(x_i) = x$. There is a natural partial ordering defined by $\pi_i \leq \pi_j$ if the covering $\pi_j : (S_j, x_j) \to (S, x)$ factors through the covering $\pi_{j,i} : (S_j, x_j) \to (S_i, x_i)$ such that $\pi_j = \pi_i \circ \pi_{j,i}$. The set of all coverings is inverse directed, and the *universal hyperbolic solenoid* $H_\infty$ is the inverse limit of this directed set (Sullivan [13]).

A point on the solenoid $H_\infty$ consists of a choice of one point $y_i \in S_i$ for each covering $\pi_i : S_i \to S$ such that there exists a fixed point $y \in S$ with $\pi_i(y_i) = y$ for each covering $\pi_i$ and that if $\pi_i \leq \pi_j$, then $\pi_{j,i}(y_j) = y_i$. The *basepoint* of $H_\infty$ is the choice of the basepoint $x_i$ on each covering surface $S_i$ over the basepoint $x$ of $S$. The topology on $H_\infty$ is the subspace topology of the product topology on the infinite product of all closed surfaces in the finite coverings of $S$.

The universal hyperbolic solenoid $H_\infty$ is a compact topological space which fibers over any surface in the finite coverings of $S$, including $S$ itself. The natural projection map $\pi_\infty : H_\infty \to S_i$ is given by projecting a point in $H_\infty$ to its $i$-th coordinate on $S_i$. The fibers are homeomorphic to a Cantor set. The path components of the solenoid $H_\infty$ are called *leaves*. Each leaf is homeomorphic to a 2-disk, and it is dense in $H_\infty$. The *base leaf* is a distinguished leaf which contains the basepoint. For details see [10].

The solenoid $H_\infty$ is locally a 2-disk times a Cantor set. An atlas on $H_\infty$ consists of a covering $\{V_i; i \in I\}$ by open sets together with the chart maps $\psi_i : V_i \to U_i \times T_i$, where $U_i$ is a 2-disk and $T_i$ is a Cantor set (called the *transversal direction*). The transition maps $\psi_j \circ \psi_i^{-1} : U_i \times T_i \to U_j \times T_j$ map the disks into the disks. A *differentiable structure* on $H_\infty$ is an atlas such that the transition maps $\psi_j \circ \psi_i^{-1}$ are $C^\infty$ in the disk direction, and they are continuous for the transversal variation in the $C^\infty$-topology on maps.

A *complex structure* on the solenoid $H_\infty$ is a choice of an atlas such that the transition maps are holomorphic in the disk direction and continuous for the transversal variation in the $C^0$-topology [13]. A complex structure on $H_\infty$ corresponds to a conformal structure. Candel [4] (see also [7]) showed that each conformal structure on $H_\infty$ contains a unique hyperbolic metric. We consider complex structures subordinate to a fixed differentiable structure. Any leaf of a complex solenoid inherits a complex structure such that it is biholomorphic to the unit disk.

A complex structure which is locally constant in the transversal direction is called a *transversely locally constant* (TLC) complex structure. A TLC complex solenoid $X$ is obtained by lifting the complex structure on a closed Riemann surface of genus greater than 1 by the natural projection map [10].

We use a different description of a TLC complex solenoid. Let $G$ be a Fuchsian uniformization group of a closed Riemann surface of genus greater than 1. Let $G_n$ be the intersection of all subgroups of $G$ of index at most $n$. Then $\{G_n\}$ is a decreasing sequence of finite index characteristic subgroups of $G$. We define a metric on $G$ by

$$\rho(A,B) = \min_{AB^{-1}\in G_n} \frac{1}{n}.$$

The closure of $G$ in the metric $\rho$ is a compact topological group homeomorphic to a Cantor set, called the *profinite completion group* $\widehat{G}$ (see [11], [3] and [12]). Let $\Delta$ denote the unit disk with the hyperbolic metric. Then $\Delta\times_G\widehat{G} := \Delta\times\widehat{G}/G$ is a TLC complex solenoid, where the action of $G$ on $\Delta\times\widehat{G}$ is defined by $A(z,t) = (Az, tA^{-1})$ (see [2] and [11]). Note that $t\in\widehat{G}$ is an equivalence class of Cauchy sequences and $tA^{-1}$ is defined by left multiplication of elements of the Cauchy sequence $t$ by $A^{-1}$. The base leaf of $\Delta\times_G\widehat{G}$ is the orbit of $\Delta\times\{id\}$ under $G$. Moreover, $\Delta\times\widehat{G}$ is the universal holomorphic covering space, and $\pi : \Delta\times\widehat{G}\to\Delta\times_G\widehat{G}$ defined by $\pi(z,t) = (z,t)/G$ is the universal holomorphic covering map for the TLC complex solenoid $\Delta\times_G\widehat{G}$. The holomorphic covering group is $G$. When restricted to a single leaf $\Delta\times\{t\}$, the covering map $\pi$ is an isometry for the corresponding hyperbolic metrics. From now on, we fix a TLC complex solenoid $H_\infty\equiv\Delta\times_G\widehat{G}$ and denote by $X$ an arbitrary complex solenoid which is not necessarily TLC.

Let $X$ and $Y$ be two complex solenoids. A homeomorphism $f : X\to Y$ is said to be *quasiconformal* if it is smooth on each leaf and continuous for the transversal variation in the $C^\infty$-topology on maps. Since $X$ and $Y$ are compact and $f$ is continuous in the $C^\infty$-topology, it follows that $f$ is quasiconformal on each leaf in the usual sense and that the Beltrami coefficient $\mu$ of $f$ is smooth, continuous for the transversal variation and $\|\mu\|_\infty < 1$.

We introduced in [12] the universal holomorphic covering for an arbitrary complex solenoid $X$ using the marking $f : H_\infty\to X$, where $f$ is a quasiconformal map and $H_\infty\equiv\Delta\times_G\widehat{G}$ is a fixed TLC complex solenoid. The *universal holomorphic covering* for $X$ is a holomorphic local isometry $\pi^X : \Delta\times T\to X$, where $T$ is the transverse part of a fixed local chart. In addition, $T$ is equipped with a homeomorphism to $\widehat{G}$ via the marking map $f : H_\infty\to X$. The covering map $\pi^X$, when restricted to a leaf $\Delta\times\{t\}$ for $t\in T$, is a conformal automorphism in the corresponding complex structures. We also introduced the covering group $G_X$ which consists of leafwise conformal isomorphisms which are continuous for the transversal variation [12]. Equivalently, $G_X$ consists of hyperbolic isometries continuous for the transversal variation. In addition, $G_X$ is isomorphic to $G$. The fundamental set for the action of $G_X$ can be chosen to be compact [12]. The marking $f : H_\infty\to X$ lifts to $\tilde{f} : \Delta\times\widehat{G}\to\Delta\times T$ such that $\pi^X\circ\tilde{f} = f\circ\pi$, and $\tilde{f}$ conjugates the action of $G$ on $\Delta\times\widehat{G}$ to the action of $G_X$ on $\Delta\times T$.

The Teichmüller space $\mathcal{T}(H_\infty)$ consists of all quasiconformal maps $f : H_\infty \to X$ up to an equivalence. Two maps $f : H_\infty \to X$ and $g : H_\infty \to Y$ are (Teichmüller) equivalent if there exists a conformal homeomorphism $c : Y \to X$ continuous for the transversal variation such that $f$ is homotopic to $c \circ g$ [13]. We denote by $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$ or by $[f] \in \mathcal{T}(H_\infty)$ the equivalence class of $f$. Equivalently, the Teichmüller space $\mathcal{T}(H_\infty)$ consists of all smooth Beltrami coefficients $\mu$ on $H_\infty$ which are continuous for the transversal variation in the $C^\infty$-topology up to an equivalence. Two Beltrami coefficients are Teichmüller equivalent if their corresponding quasiconformal maps are equivalent as above. We denote by $[\mu] \in \mathcal{T}(H_\infty)$ the equivalence class of $\mu$.

## 3. Homotopies and isotopies

Let $f$ and $g$ be two homotopic quasiconformal maps of a Riemann surface $S$ onto itself. Then $f$ and $g$ are isotopic through quasiconformal maps. This is a classical result when $S$ is a closed surface. For arbitrary Riemann surfaces this was proved by Earle and McMullen [6] using the barycentric extension [5].

We consider the similar question for solenoids, namely whether the homotopy class of a quasiconformal self-map of a complex solenoid $X$ is the same as the corresponding isotopy class. We require that the homotopies and the isotopies are leaf preserving. An isotopy $f(p, s)$, $p \in X, 0 \leq s \leq 1$, is *uniformly quasiconformal* if there exists $K > 1$ such that each $f(p, s)$ is $K$-quasiconformal for every $s$.

We start by considering the homotopy class of the identity map.

**Lemma 3.1.** *Let $f : X \to X$ be a quasiconformal self-map of a complex solenoid $X$ homotopic to the identity map. Then $f$ is isotopic to the identity map by a uniformly quasiconformal isotopy.*

*Proof.* Assume first that $X \equiv \Delta \times_G \widehat{G}$ is a TLC complex solenoid and $f$ a quasiconformal self-map of $X$. Since $f$ is homotopic to the identity map there exists a homotopy $f_s(p)$, $p \in X, 0 \leq s \leq 1$, such that $f_s(p)$ is continuous in $p$ and $s$, and

$$f_0 = id, f_1 = f.$$

Now, fix a point $p_0 \in X$. Since the path $f_s(p_0)$, $0 \leq s \leq 1$, connects the points $p_0$ and $f(p_0)$, we have that these two points lie in the same path component in $X$. Since the leaves are the path components in $X$, the conclusion is that the map $f$ preserves each leaf in $X$. Also, let $d(p) = \sigma(p, f(p))$, where $\sigma$ is the hyperbolic metric on the leaf that contains points $p, f(p)$. We just saw that this is well defined, and from the continuity of $f_s(p)$ it follows that $d(p)$ is a continuous function on $X$. Since $X$ is a compact topological space, there exists $M > 0$ such that $d(p) < M$ for $p \in X$. Therefore, the restriction of $f$ on each leaf is a quasiconformal self-map of that leaf which is a finite hyperbolic distance apart for the identity map. So, we have that $f$ extends continuously to the identity map on the ideal boundary of each leaf.

Using the facts that $f$ is quasiconformal and that $f$ preserves each leaf, it follows that the restriction $\bar{f}$ of $f$ to the base leaf of $\Delta \times_G \widehat{G}$ conjugates a finite index subgroup $H$ of $G$ onto another finite index subgroup $K$ (Odden [11]). Since $\bar{f}$ is the identity on the boundary of the base leaf, it has to conjugate each element of $H$ onto itself. We identify the base leaf with the unit disk $\Delta$. Then $X \equiv \Delta \times_H \widehat{H}$ and $F : \Delta \times \widehat{H} \to \Delta \times \widehat{H}$, given by $F(z, t) = (\bar{f}(z), t)$, is the lift of $f : X \to X$ to the universal covering $\pi : \Delta \times \widehat{H} \to X$ invariant under the action of $H$. Earle and McMullen [6] showed that there exists an isotopy $\bar{f}_s$, $0 \leq s \leq 1$, such that $\bar{f}_1 = \bar{f}$ and $\bar{f}_0 = id$, and which is invariant under the action of $H$. Then $F_s : \Delta \times \widehat{H} \to \Delta \times \widehat{H}$, given by $F_s(z,t) = (\bar{f}_s(z), t)$, is invariant under the action of $H$ on $\Delta \times \widehat{H}$, and it projects to leaf preserving, quasiconformally bounded, isotopy $f_s : X \to X$ such that $f_1 = f$ and $f_0 = id$. We note that the method of Earle and McMullen provides that maps $f_s$ in the isotopy are smooth if $f$ is smooth.

Assume that $X$ is a complex solenoid which is not TLC. Let $g : H_\infty \to X$ be a quasiconformal map, where $H_\infty$ is a TLC complex solenoid. The quasiconformal map $g^{-1} \circ f \circ g : H_\infty \to H_\infty$ is homotopic to the identity. From the above, it follows that $g^{-1} \circ f \circ g$ is isotopic to the identity. Conjugating this isotopy back to $X$ we see that $f$ is isotopic to the identity as well. □

Let $f : X \to Y$ and $g : X \to Y$ be two homotopic quasiconformal maps. Then $g^{-1} \circ f : X \to X$ is homotopic to the identity. By the above theorem, the map $g^{-1} \circ f$ is isotopic to the identity. This proves the following theorem.

**Theorem 3.1.** *Let $f : X \to Y$ and $g : X \to Y$ be two homotopic quasiconformal maps between complex solenoids $X$ and $Y$. Then $f$ and $g$ are isotopic by leaf preserving uniformly quasiconformal isotopy.*

Any homeomorphism between two closed Riemann surfaces is homotopic (even isotopic) to a quasiconformal map. In general, two homeomorphic Riemann surfaces are not quasiconformally equivalent, e.g. a punctured disk and an annulus. Let $h : H_\infty \to H_\infty$ be a homeomorphism of a TLC complex solenoid $H_\infty \equiv \Delta \times_G \widehat{G}$. We show that it is homotopic to a quasiconformal self-map of $H_\infty$.

**Theorem 3.2.** *Let $h : H_\infty \to H_\infty$ be a base leaf preserving homeomorphism of a TLC complex solenoid onto itself. Then there exists a quasiconformal map $f : H_\infty \to H_\infty$ which is homotopic to $h$.*

*Proof.* Since $h$ is base leaf preserving, it conjugates the action of a finite subgroup $H$ of $G$ onto the action of a finite subgroup $K$ of $G$ on the base leaf (see [11] and [3]). Therefore, $h$ descends to a homeomorphism $\bar{h}$ between closed Riemann surfaces $\Delta/H$ and $\Delta/K$. As such, $\bar{h}$ is homotopic to a quasiconformal map $\bar{f} : \Delta/H \to \Delta/K$. Consequently $\bar{f}$ lifts to a quasiconformal map $f$ of the base leaf onto itself homotopic to $h$. The map $f$ produces a quasiconformal map of $H_\infty$ onto itself by extending it in a locally constant fashion in the transverse direction. □

## 4. The stability property

A family of $K$-quasiconformal maps of the complex solenoid $H_\infty$ onto a fixed complex solenoid $X$ does not have to be compact (as far as we know the first examples of this were given by Adam Epstein). This is in contrast to the compactness properties of quasiconformal maps on Riemann surfaces. Thus, some aspects of the issues related to convergence of quasiconformal maps of solenoids may be more subtle than in the case of Riemann surfaces.

On the other hand, let $f_n : S \to S$ be a sequence of homeomorphisms of a closed Riemann surface $S$ onto itself which converges uniformly on $S$ to the identity map. It is well known that in this case there exists $n_0$ such that each $f_n$ for $n > n_0$ is homotopic to the identity map (the stability property for closed Riemann surfaces). This property is not true for certain open Riemann surfaces (e.g. the unit disc) or geometrically infinite Riemann surfaces (e.g. Riemann surfaces of infinite genus). The complex solenoid $H_\infty$ is a compact space, but each leaf is conformally equivalent to the unit disk. Therefore, it is interesting to investigate whether the corresponding stability property holds in this case.

First, we consider the case when $H_\infty$ is endowed with a TLC complex structure $H_\infty \equiv \Delta \times_G \widehat{G}$, where $G$ is the Fuchsian group uniformizing a given closed Riemann surface. Recall that $\pi : \Delta \times \widehat{G} \to H_\infty$ is the universal covering. We identify the base leaf of $H_\infty$ with the unit disk $\Delta$ using the covering map $\pi : \Delta \equiv \Delta \times \{id\} \to \pi(\Delta \times \{id\}) \subset H_\infty$. The following theorem gives a version of the stability property for homeomorphisms of TLC solenoids.

**Theorem 4.1.** *Let $H_\infty$ be a TLC complex solenoid and let $f_n : H_\infty \to H_\infty$ be a sequence of base leaf preserving quasiconformal self-maps of $H_\infty$ which converges uniformly to the identity map in the topology of $H_\infty$. Then there exists $n_0$ such that $f_n$ is homotopic to a base leaf preserving conformal self-map $c_n : H_\infty \to H_\infty$, for all $n > n_0$.*

*Proof.* The restriction of the topology of the solenoid $H_\infty$ to the base leaf $\Delta \times \{id\}$ is given as follows. Let $\sigma(z, w)$ be the hyperbolic distance in the unit disk $\Delta$. Then the distance $d(z, w) = \inf_{A \in G} \max\{\sigma(z, Aw), \rho(A, id)\}$ induces the subspace topology on the base leaf $\Delta$ (see [11]) (recall that $\rho(A, id)$ is the distance function for the group $G$). Let $\mathbf{B}_\delta(x)$ be the ball of radius $\delta$ around $x \in \Delta$ for the metric $d$ introduced above (the base leaf is identified with $\Delta$). Since $f_n$ converges uniformly in the metric $d$ to the identity map, for a given $\epsilon > 0$, there exist $n_0 > 0$ and $\delta > 0$ such that

$$f_n(\mathbf{B}_\delta(x)) \subset \mathbf{B}_\epsilon(x), \tag{4.1}$$

for all $x \in \Delta$ and for all $n > n_0$. We choose $\epsilon > 0$ to be smaller than the minimum of the translation lengths of the elements of $G - \{id\}$ (since $G$ is a covering group of a closed Riemann surface, this minimum is attained and is positive). Fix $n > n_0$. By (4.1) and from the definition of the metric $d$, we have that for each $x \in \Delta$ there exists a unique $B_k^x \in G_k$, $k = k(x, n)$, such that

$$\sigma(B_k^x \circ f_n(x), x) < \epsilon \tag{4.2}$$

and $\frac{1}{k} < \epsilon$.

The set $\Delta(A, \epsilon) = \{x \in \Delta | \sigma(A \circ f_n(x), x) < \epsilon\}$ is open in $\Delta$ and $\Delta(A, \epsilon) \cap \Delta(B, \epsilon) = \emptyset$ whenever $A, B \in G$ and $A \neq B$, by our choice of $\epsilon$. By (4.2) we get that $\bigcup_{x \in \Delta} \Delta(B_k^x, \epsilon) = \Delta$. Since the intersection of two different elements in the union is empty, we conclude that $B_k^x$ is constant in $x$. Namely, there exists a unique element $B_k = B_{k(n)} \in G$ such that $\sigma(B_k \circ f_n(x), x) < \epsilon$, for all $x \in \Delta$. Thus the map $f_n$, when restricted to the base leaf, is homotopic to $A_k = B_k^{-1}$.

The map $A_k : \Delta \to \Delta$ of the base leaf onto itself extends to $\tilde{\tilde{A}}_k : \Delta \times \widehat{G_k} \to \Delta \times \widehat{G_k}$ by the formula $\tilde{\tilde{A}}_k(z, t) = (A_k(z), A_k t A_k^{-1})$. The extension satisfies $\tilde{\tilde{A}}_k \circ B(z, t) = (A_k B A_k^{-1}) \tilde{\tilde{A}}_k(z, t)$, $B \in G$, and it projects to the base leaf preserving conformal map $c_n := \hat{A}_k : H_\infty \to H_\infty$. $\square$

The previous theorem shows that in the case of TLC solenoids the uniform convergence of a sequence of quasiconformal maps implies that all of them (except maybe finitely many of them) are homotopic to each other modulo a conformal automorphism of $H_\infty$. The following theorem gives the appropriate statement for the general case.

**Theorem 4.2.** *Let $X$ be a non-TLC complex solenoid with the base leaf and let $f_n : X \to X$ be a sequence of base leaf preserving quasiconformal maps uniformly converging to the identity. Then, for arbitrary TLC complex solenoid $Y$, there exists $n_0$ such that $f_n$ is isotopic to a conjugate of a conformal self-map of $Y$, for $n > n_0$.*

*Proof.* Let $f : Y \to X$ be an arbitrary quasiconformal map. Let $H$ be a Fuchsian uniformizing group of $Y$, namely $Y \equiv \Delta \times_H \widehat{H}$. We arrange that the canonical baseleaf of $Y$ is mapped to the baseleaf of $X$ by possibly precomposing $f$ with a conformal transversal motion on $Y$. Then $g_n := f^{-1} \circ f_n \circ f : Y \to Y$ is a quasiconformal self-map of $Y$ which converges uniformly to the identity. Consequently $g_n$ is homotopic to a conformal self-map $c_n$ of $Y$ by Theorem 4.1. It follows that $f_n$ is homotopic to $f \circ c_n \circ f^{-1}$. □

A sequence of quasiconformal self-maps of a non-TLC solenoid which uniformly converges to the identity is not necessarily homotopic to a sequence of conformal maps. This is a consequence of the proof of Theorem 5.1 below (put $t_0 = id$ in the proof). Therefore, Theorem 4.2 is the strongest statement for non-TLC solenoids.

## 5. Orbits of $MCG_{BLP}(H_\infty)$

Let $h : H_\infty \to H_\infty$ be a quasiconformal self-map of $H_\infty$. Then the map $h$ induces the *geometric isomorphism* $\rho_h : \mathcal{T}(H_\infty) \to \mathcal{T}(H_\infty)$ by the formula $\rho_h([f]) = [f \circ h^{-1}]$, for $[f] \in \mathcal{T}(H_\infty)$. The geometric isomorphism $\rho_h$ is a biholomorphic isometry of the Teichmüller space $\mathcal{T}(H_\infty)$. We do not require that $h$ preserves any leaf of $H_\infty$. Any other quasiconformal map $h_1 : H_\infty \to H_\infty$ which is homotopic (isotopic) to $h$ induces the same geometric isomorphism as $h$, namely $\rho_{h_1} = \rho_h$. The group of all quasiconformal self-maps of $H_\infty$, up to homotopies (isotopies), is called the *mapping class group* $MCG(H_\infty)$ of $H_\infty$. An element of the mapping class group $MCG(H_\infty)$ induces a well-defined *geometric* isomorphism of $\mathcal{T}(H_\infty)$ as above.

We define the *base leaf preserving mapping class group* $MCG_{BLP}(H_\infty)$ of $H_\infty$ to consist of all base leaf preserving, quasiconformal self-maps of $H_\infty$ up to isotopy. Recall that $H_\infty \equiv \Delta \times_G \widehat{G}$. If $h : H_\infty \to H_\infty$ preserves the base leaf, then it induces a partial automorphism of $G$ (see C. Odden's thesis [11] or [2]). Namely, there exist two finite index subgroups $H$ and $K$ of $G$ such that the restriction of $h$ to the base leaf of $H_\infty$ conjugates $H$ to $K$. Two partial automorphisms of $G$ are *equivalent* if they agree on a finite index subgroup of $G$. The equivalence class of a partial automorphism is called the *virtual automorphism*. The composition of any two virtual automorphisms is well defined. The set of virtual automorphisms has the group structure with the group operation being composition. Finally, the group $MCG_{BLP}(H_\infty)$ of the base leaf preserving quasiconformal self-maps of $H_\infty$ up to isotopy is naturally isomorphic to the group of virtual automorphisms of $G$ (see [11] or [2]).

The classical conjecture of Ehrenprise concerning finite covers of closed Riemann surfaces (see [2]) is equivalent to the statement that the action of $MCG_{BLP}(H_\infty)$ on $\mathcal{T}(H_\infty)$ is topologically effective, namely each orbit under $MCG_{BLP}(H_\infty)$ of a point is dense. We show below that $MCG_{BLP}(H_\infty)$ does not act properly discontinuously on $\mathcal{T}(H_\infty)$.

**Theorem 5.1.** *There exists a dense subset of $\mathcal{T}(H_\infty)$ such that the orbit of the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ of any point in this subset has accumulation points in $\mathcal{T}(H_\infty)$.*

*Proof.* Let $A \in G$. Define $\tilde{\hat{A}} : \Delta \times \widehat{G} \to \Delta \times \widehat{G}$ by

$$\tilde{\hat{A}}(z,t) = (Az, AtA^{-1}).$$

The expression $AtA^{-1}$ is well defined because, by the definition, $t \in \widehat{G}$ is an equivalence class of Cauchy sequences in $G$ and so is $AtA^{-1}$. Note that

$$\tilde{\hat{A}}(B(z,t)) = (ABz, AtB^{-1}A^{-1}) = (ABA^{-1})(Az, AtA^{-1}) = (ABA^{-1})(\tilde{\hat{A}}(z,t)).$$

Thus $\tilde{\hat{A}}(z,t)$ projects to the isometry $\hat{A} : H_\infty \to H_\infty$. The map $\tilde{\hat{A}}$ differs from $A$ on $\Delta \times \widehat{G}$. In particular, the map $A : \Delta \times \widehat{G} \to \Delta \times \widehat{G}$ projects to the identity on $H_\infty$, whereas $\hat{A}$ is not the identity as long as $A \neq id$. In addition, $\hat{A} : H_\infty \to H_\infty$ preserves the base leaf.

Let $\tilde{\mu}$ be the Beltrami coefficient on $\Delta \times \widehat{G}$ which arises as a lift of a Beltrami coefficient on $H_\infty$; namely $A^*\tilde{\mu} = \tilde{\mu}$ for all $A \in G$, where $A^*\tilde{\mu}(z,t) = \tilde{\mu}(Az, tA^{-1})\frac{\overline{A'(z)}}{A'(z)}$. Then

$$(\tilde{\hat{A}})^*(\tilde{\mu})(z,t) = \tilde{\mu}(A(z), AtA^{-1})\frac{\overline{A'(z)}}{A'(z)} = (A^*\tilde{\mu})(z, At) = \tilde{\mu}(z, At)$$

by the invariance of $\tilde{\mu}$ under the action of $G$ on $\Delta \times \widehat{G}$. In other words, $(\tilde{\hat{A}})^*(\tilde{\mu})$ on $\Delta \times \{t\}$ is obtained by the pullback of $\tilde{\mu}$ from the leaf $\Delta \times \{At\}$ via the transverse identification map $(z,t) \mapsto (z, At)$.

Let $\mu$ be a smooth Beltrami coefficient on $H_\infty$ which is not Teichmüller equivalent to a TLC Beltrami coefficient and let $\tilde{\mu}$ be the lift of $\mu$ to $\Delta \times \widehat{G}$. Then there exists $t_0 \in \widehat{G} - G$ such that $\tilde{\mu}(\cdot, id)$ is not equivalent to $\tilde{\mu}(\cdot, t_0)$ when considered as Beltrami coefficients on the unit disk $\Delta (\equiv \Delta \times \{id\} \equiv \Delta \times \{t_0\})$. If not, then $\tilde{\mu}$ would be constant everywhere by the density of $\widehat{G} - G$ in $\widehat{G}$. Furthermore, there is a sequence $A_n \in G$ such that $\rho(A_n, t_0) \to 0$ as $n \to \infty$ and such that $\tilde{\mu}(\cdot, A_n)$ is not Teichmüller equivalent to $\tilde{\mu}(\cdot, t_0)$. If this was not true, then $\tilde{\mu}(\cdot, t)$ would be a TLC Beltrami coefficient in a neighborhood of $t_0 \in \widehat{G}$. Consequently, $\mu$ would be a TLC smooth Beltrami coefficient on $H_\infty$ which contradicts the choice of $\mu$.

Sullivan [13] showed that the restriction of the complex structure on $H_\infty$ to the base leaf uniquely determines the complex structure on $H_\infty$. Note that $\tilde{\mu}(z, A_n) \to \tilde{\mu}(z, t_0)$ as $n \to \infty$ in the essential supremum norm because of the transverse continuity of $\tilde{\mu}$ on the fundamental set and because of its invariance under $G$. Since $(\tilde{\hat{A}}_n)^*(\tilde{\mu})(z, id) = \tilde{\mu}(z, A_n)$ we get that the orbit $(\tilde{\hat{A}}_n)^*(\tilde{\mu})$ converges to a smooth Beltrami coefficient $\tilde{\nu}(z,t) := \tilde{\mu}(z, t_0 t)$. Therefore, $(\hat{A}_n)^*(\mu) \to \nu$ as $n \to \infty$, where $\nu$ is the projection of $\tilde{\nu}$ on $H_\infty$. We showed that for any non-TLC point $[\mu] \in \mathcal{T}(H_\infty)$ there exists a sequence $A_n \in G$ such that $(\hat{A}_n)^*(\mu)$ has an accumulation point, where $\hat{A}_n \in MCG_{BLP}(H_\infty)$. $\square$

A direct consequence of the above theorem as well as Theorem 7.1 is the following.

**Corollary 5.1.** *The base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ does not act properly discontinuously on $\mathcal{T}(H_\infty)$.*

Let $h : H_\infty \to H_\infty$ be a quasiconformal map which does not preserve the base leaf. We show that $[h] \in MCG(H_\infty)$ can be written as a composition of an isometry of $H_\infty$ which does not preserve the base leaf and an element of $MCG_{BLP}(H_\infty)$.

**Theorem 5.2.** *Any $[h] \in MCG(H_\infty) - MCG_{BLP}(H_\infty)$ can be written as the composition of an isometry of $H_\infty$ which does not preserve the base leaf and an element of $MCG_{BLP}(H_\infty)$. In addition, each element of $MCG(H_\infty)$ which does not preserve the base leaf can be obtained as the limit of a sequence of the base leaf preserving elements.*

*Proof.* Consider $h : \Delta \times_G \widehat{G} \to \Delta \times_G \widehat{G}$ and let $l$ be the image of the base leaf under $h$. By our assumption, $l$ is not the base leaf. The preimage of $l$ under the universal covering map $\pi : \Delta \times \widehat{G} \to \Delta \times_G \widehat{G}$ consists of countably many leaves. We fix $t_0 \in \widehat{G}$ such that $\pi(\Delta \times \{t_0\}) = l$. Define $\tilde{\bar{t}}_0 : \Delta \times \widehat{G} \to \Delta \times \widehat{G}$ by $\tilde{\bar{t}}_0(z,t) = (z, t_0 t)$. Then $\tilde{\bar{t}}_0$ projects to an isometry $\bar{t}_0$ of $H_\infty$ which sends the base leaf to $l$ and $h_0 = h \circ \bar{t}_0^{-1} : H_\infty \to H_\infty$ preserves the base leaf. Consequently, $h = h_0 \circ \bar{t}_0$, where $h_0 \in MCG_{BLP}(H_\infty)$.

Let $A_n \in G$ be such that $A_n \to t_0$ as $n \to \infty$. Then $\bar{A}_n \to \bar{t}_0$ by the proof of Theorem 5.1. Thus $h_0 \circ \bar{A}_n \in MCG_{BLP}(H_\infty)$ and $h_0 \circ \bar{A}_n \to h_0 \circ \bar{t}_0 = h$ as $n \to \infty$. □

## 6. Finite subgroups of $MCG_{BLP}(H_\infty)$

A finite subgroup of the mapping class group of a closed Riemann surface $S$ is realized as an isotropy subgroup of a point in the Teichmüller space $\mathcal{T}(S)$ (see [8]). Let $f : S \to S$ be a smooth quasiconformal self-map of the closed Riemann surface $S$. Let $G$ be a Fuchsian group such that $S = \Delta/G$ and let $\tilde{f} : \Delta \to \Delta$ be the lift of $f$ which conjugates $G$ onto itself. The map $\tilde{F} : \Delta \times \widehat{G} \to \Delta \times \widehat{G}$ defined by $\tilde{F}(z,t) = (\tilde{f}(z), \tilde{f} t \tilde{f}^{-1})$ is invariant under the action of $G$ on $\Delta \times \widehat{G}$, where $\tilde{f} t \tilde{f}^{-1}$ is obtained by conjugating elements of a Cauchy sequence representing $t$. Note that $\tilde{f} t \tilde{f}^{-1}$ is a Cauchy sequence for $\rho$ because the groups $G_n$ are characteristic. Therefore $\tilde{F}$ projects to a quasiconformal map $F : H_\infty \equiv \Delta \times_G \widehat{G} \to H_\infty$, which is locally constant in the transverse direction. Such a map $F : H_\infty \to H_\infty$ is said to be *mapping class-like* (see [11]). Note that a single $f : S \to S$ induces infinitely many corresponding maps $F : H_\infty \to H_\infty$ depending on the lift $\tilde{f} : \Delta \to \Delta$.

Let $F_{mod}$ be a finite subgroup of the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$ of the solenoid $H_\infty$. We show that $F_{mod}$ consists only of mapping class-like elements of $MCG_{BLP}(H_\infty)$. □

**Theorem 6.1.** *Assume that $F_{mod}$ is a finite subgroup of the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$. Then $F_{mod}$ is mapping class like. In addition, the group $F_{mod}$ is realized as a subgroup of the isotropy group of the solenoid with a TLC complex structure.*

*Proof.* We use an idea of C. Odden [11]. Let $\tilde{f}$ be the restriction of the lift of an element of $F_{mod}$ to the base leaf $\Delta \times \{id\} \subset \Delta \times \widehat{G}$. Then $\tilde{f}$ conjugates a finite index subgroup $G_{\tilde{f}}$ of $G$ onto another finite index subgroup of $G$ (see [11] and [3]).

We define

$$H_1 = \bigcap_{\tilde{f} \in F_{mod}} G_{\tilde{f}}.$$

Since $H_1$ is a finite intersection of finite index subgroups of $G$, it follows that $H_1$ has a finite index in $G$. We define a finite index subgroup

$$H = \bigcap_{\tilde{f} \in F_{mod}} \tilde{f} \circ H_1 \circ (\tilde{f})^{-1}$$

of $G$. By the definition $H$ is invariant under conjugations by all elements $\tilde{f} \in F_{mod}$. Consequently, each $\tilde{f} \in F_{mod}$ projects to a self-map of the Riemann surface $\Delta/H$. Thus, elements of $F_{mod}$ are lifts of self-maps of the closed surface $\Delta/H$. Namely, $F_{mod}$ consists of mapping class-like elements.

By the Nielsen realization property for closed surfaces [8], there exists a Fuchsian group $H_2$ which is quasiconformally conjugate to $H$ and such that $\tilde{f} \in F_{mod}$ are isotopic to the lifts of isometries of the closed Riemann surface $\Delta/H_2$. Then, the solenoid $\Delta \times_{H_2} \widehat{H_2}$ with the TLC complex structure obtained by the lift of the complex structure on $\Delta/H_2$ has $F_{mod}$ as a subgroup of its isometry group. Therefore $F_{mod}$ is the isotropy group of $[g : H_\infty \to \Delta \times_{H_2} \widehat{H_2}] \in \mathcal{T}(H_\infty)$, where $g$ is a quasiconformal map. $\square$

By the above theorem, the restriction of a finite subgroup of $MCG_{BLP}(H_\infty)$ to the base leaf of the TLC complex solenoid fixed by the group consists of conformal automorphisms for the complex structure of the base leaf (which is conformally equivalent to the unit disk). A finite subgroup of the orientation preserving automorphism group of the disk is necessarily cyclic with an elliptic generator. Therefore, any finite subgroup of $MCG_{BLP}(H_\infty)$ is necessarily cyclic.

There exists a finite order element $F$ of the mapping class group $MCG(S)$ of a closed Riemann surface $S$ which lifts to a finite order element of $MCG_{BLP}(H_\infty)$. To see this, let $S_1$ be a Riemann surface quasiconformally equivalent to $S$ such that $F : S_1 \to S_1$ is isotopic to a conformal map $c : S_1 \to S_1$. Such $S_1$ and $c$ exist by the Nielsen Realization property for closed surfaces (see [8]). Then $c$ lifts to a conformal map $C : X \to X$, where the TLC complex solenoid $X$ is obtained by lifting the complex structure of $S_1$. The lift $C$ is conformal on leaves of $X$. Clearly, $C$ has finite order if and only if it is elliptic on the baseleaf. In that case, $C$ fixes a point on the baseleaf and therefore its projection $c : S_1 \to S_1$ has a fixed point. On the other hand, if $c : S_1 \to S_1$ has a fixed point, then there exists a lift $\tilde{c} : \Delta \to \Delta$ which fixes a point on the base leaf $\Delta$. Then the holomorphic extension map $C : X \to X$ of $\tilde{c}$ fixes a point on the baseleaf. However, not every lift $\tilde{c} : \Delta \to \Delta$ has a fixed point even though $c : S_1 \to S_1$ has a fixed point. This shows that finite subgroups of $MCG_{BLP}(H_\infty)$ exist.

**Corollary 6.1.** *There exists a non-trivial finite subgroup of $MCG_{BLP}(H_\infty)$. Any finite subgroup of $MCG_{BLP}(H_\infty)$ is necessarily cyclic.*

The following is a direct corollary to the above theorem.

**Corollary 6.2.** *Elements of $MCG_{BLP}(H_\infty)$ which are not mapping class-like are of infinite order.*

## 7. Isotropy subgroups of $MCG_{BLP}(H_\infty)$

We keep the assumption that $H_\infty = \Delta \times_G \widehat{G}$, where $G$ is a Fuchsian uniformizing group of a closed Riemann surface of genus greater than 1. The commensurator $Comm(G)$ of $G$ in the automorphism group $Aut(\Delta)$ of the unit disk $\Delta$ consists of all $A \in Aut(\Delta)$ such that $AGA^{-1} \cap G$ is of finite index in both $AGA^{-1}$ and $G$. Chris Odden [11] proved that the isotropy subgroup of the base point of $\mathcal{T}(H_\infty)$ is isomorphic to the commensurator of $G$ (see also [3]). The group $G$ is a subgroup of the commensurator $Comm(G)$. The classical result of Margulis [9] states that there exist only countably many groups whose commensurators are dense in $Aut(\Delta)$. These are arithmetic groups, and they induce TLC complex structures on the solenoid which have large isometry groups. On the other hand if the commensurator is not dense in $Aut(\Delta)$, then the group $G$ has the finite index in $Comm(G)$.

Assume now that $X$ is a solenoid with a non-TLC complex structure. We recall that $\Delta \times T$ is the universal covering space and that the covering group $G_X$ is isomorphic to $G$. An element $A_X \in G_X$ is a leafwise isometry on $\Delta \times T$ which is continuous for the transversal variation. We define the *automorphism group* $Aut(\Delta \times T)$ of $\Delta \times T$ to consist of all leafwise isometries which are continuous for the transversal variation. Clearly, $G_X$ is a subgroup of $Aut(\Delta \times T_X)$.

The *commensurator* $Comm(G_X)$ of $G_X$ consists of all transversely continuous leafwise isometries $A_X : \Delta \times T_1 \to \Delta \times T_2$ such that $T_1 = \widehat{H_X}$ and $T_2 = \widehat{K_X}$ for two finite index subgroups $H_X$ and $K_X$ of $G_X$ with $A_X H_X A_X^{-1} = K_X$ up to an equivalence; $A_X$ and $B_X$ are equivalent if they agree on the intersection of their domains. We show that $Comm(G_X)$ is isomorphic to the isotropy subgroup of the point $[f : H_\infty \to X] \in \mathcal{T}(H_\infty)$.

**Theorem 7.1.** *Let $f : H_\infty \to X$ be a quasiconformal map, where $X$ is an arbitrary non-TLC complex solenoid. Then the isotropy subgroup of $[f] \in \mathcal{T}(H_\infty)$ in $MCG_{BLP}(H_\infty)$ is isomorphic to the commensurator group $Comm(G_X)$ of the covering group $G_X$ for $X$.*

*Proof.* Let $A_X \in Comm(G_X)$. Then there exist two finite index subgroups $H_X$ and $K_X$ of $G_X$ such that $A_X H_X A_X^{-1} = K_X$. Let $\tilde{f} : \Delta \times \widehat{G} \to \Delta \times T$ be the lift of $f : H_\infty \to X$ such that $\tilde{f}$ conjugates the action of $G$ to the action of $G_X$ and gives the identification $\tilde{f} : \widehat{G} \equiv T$ (see [12]). Denote by $H$ and $K$ finite index subgroups of $G$ such that $\tilde{f}$ conjugates them into $H_X$ and $K_X$, respectively. Let $T^1 \subset T$ with $T^1 \equiv \widehat{H}$ and let $T^2 \subset T$ with $T^2 \equiv \widehat{K}$, where the identifications are the restrictions of the identification $T \equiv \widehat{G}$. Then $\Delta \times_{H_X} T^1 \equiv \Delta \times_{K_X} T^2 \equiv X$. Note that any $t \in T^1 \equiv \widehat{H}$ is an equivalence class of Cauchy sequences in $H_X \equiv H$, where the identification of $H$ with $H_X$ is by the conjugation with $\tilde{f}$.

We define $\tilde{\hat{A}}_X : \Delta \times T^1 \to \Delta \times T^2$ by $\tilde{\hat{A}}_X(z,t) = (A_X(z), A_X t A_X^{-1})$, where $A_X t A_X^{-1}$ is defined by conjugating elements of the Cauchy sequence corresponding to $t \in T^1 \equiv \widehat{H}$ with $A_X \in Comm(G_X)$ and where $A_X(z)$ is the $z$-variable part of $A_X(z,t)$. The sequence $A_X t A_X^{-1}$ is Cauchy in $K_X$ because $A_X H_X A_X^{-1} = K_X$. Similar to the proof of Theorem 4.1, $\tilde{\hat{A}}_X$ projects to a base leaf preserving isometry

$$\hat{A}_X : \Delta \times_{H_X} T^1 \to \Delta \times_{K_X} T^2.$$

Clearly, $\widehat{(A \circ B)}_X = \hat{A}_X \circ \hat{B}_X$. The map $I : A_X \mapsto \hat{A}_X$ from $Comm(G_X)$ into the group of isometries $Isom(X)$ for the hyperbolic metric on $X$ is a homomorphism. We note that $Isom(X)$ is exactly the isotropy group of $[f] \in \mathcal{T}(H_\infty)$.

We show that $I$ is injective. It is enough to show that if $\hat{A}_X$ acts by the identity on $X$, then $A_X = id$ for the equivalence relation on $Comm(G_X)$. The assumption that $\hat{A}_X$ is the identity implies the existence of $B_X \in K_X$ such that $\tilde{\hat{A}}_X(z,t) = B_X(z,t)$ for all $(z,t) \in \Delta \times T_1$. If we let $t = id$ above, this implies $B_X = id$. Since $A_X H_X A_X^{-1} = K_X$ and the base leaf is dense in $H_\infty$, it follows that $\tilde{\hat{A}}_X$ equals the identity on $\Delta \times \widehat{H_X}$. Therefore, $I$ is one to one.

It remains to show that $I$ is surjective. Let $\psi : X \to X$ be a base leaf preserving isometry of $X$. Then $\psi_1 = f^{-1} \circ \psi \circ f : H_\infty \to H_\infty$ is a base leaf preserving quasiconformal map. We identify the base leaf with $\Delta$. The restriction $\tilde{\psi}_1$ of $\psi_1$ to the base leaf conjugates a finite index subgroup $H$ onto another finite index subgroup $K$ of the group $G$ (see [11] or [3]).

Define $\tilde{\Psi}_1 : \Delta \times \widehat{H} \to \Delta \times \widehat{K}$ by $\tilde{\Psi}_1(z,t) = (\tilde{\psi}_1(z), \tilde{\psi}_1 t \tilde{\psi}_1^{-1})$, where $\tilde{\psi}_1(z)$ is constant in $t$ and $\tilde{\psi}_1 t \tilde{\psi}_1^{-1}$ is an element of $\widehat{K}$ (see [11]). We note that $\tilde{\Psi}_1$ projects to $\Psi_1 : \Delta \times_H \widehat{H} \to \Delta \times_K \widehat{K}$ because

$$(\tilde{\Psi}_1 \circ A)(z,t) = (\tilde{\psi}_1(Az), \tilde{\psi}_1 t A^{-1} \tilde{\psi}_1^{-1}) = (B(\tilde{\psi}_1(z)), \tilde{\psi}_1 t \tilde{\psi}_1^{-1} B^{-1}) = (B \circ \tilde{\Psi}_1)(z,t)$$

for $A \in H$ and $\tilde{\psi}_1 \circ A \circ \tilde{\psi}_1^{-1} = B \in K$. Further, $\Psi_1$ is equal to $\psi_1$ because $\Psi_1$ agrees with $\psi_1$ on the base leaf. Thus $\tilde{\Psi}_1 : \Delta \times \widehat{H} \to \Delta \times \widehat{K}$ is a lift of $\psi_1 : H_\infty \to H_\infty$ and $\tilde{\Psi}_1$ conjugates the action of $H$ on $\Delta \times \widehat{H}$ to the action of $K$ on $\Delta \times \widehat{K}$. Consequently, $\tilde{\Psi} := \tilde{f} \circ \tilde{\Psi}_1 \circ \tilde{f}^{-1} : \Delta \times T^1 \to \Delta \times T^2$ conjugates the action of $H_X$ on $\Delta \times T^1$ to the action of $K_X$ on $\Delta \times T^2$. Then $\tilde{\Psi}$ projects to a self-map $\Psi$ of $X$ and $\Psi$ agrees with the isometry $\psi : X \to X$ on the base leaf. Therefore $\Psi$ is an isometry, and its lift $\tilde{\Psi} : \Delta \times T^1 \to \Delta \times T^2$ is an isometry itself. □

It is obvious that $G_X$ is a subgroup of $Comm(G_X)$. Thus any non-TLC complex solenoid $X$ has an infinite group of isometries $Isom(X)$. The same property holds for TLC solenoids (see [11] or [2]).

**Corollary 7.1.** *Any point in $\mathcal{T}(H_\infty)$ has an infinite isotropy subgroup in the base leaf preserving mapping class group $MCG_{BLP}(H_\infty)$.*

In the view of the corresponding situation for TLC solenoids we ask the following question:

**Question.** *Is there a non-TLC solenoid $X$ such that the restriction of the base leaf preserving isometry group $Isom(X)$ to the base leaf of $X$ is dense in $Aut(\Delta)$?*

We showed (see Corollary 7.1) that each point in $\mathcal{T}(H_\infty)$ has an infinite isotropy subgroup in $MCG_{BLP}(H_\infty)$. By the counting argument we conclude that some elements of $MCB_{BLP}(H_\infty)$ have uncountably many fixed points. We show that $\hat{A} \in MCG_{BLP}(H_\infty)$, for $A \in G$, has uncountably many fixed points:

**Proposition 7.1.** *Let $H_\infty = \Delta \times_G \widehat{G}$ be a TLC complex solenoid, where $G$ is a Fuchsian group. Then $\hat{A} : \mathcal{T}(H_\infty) \to \mathcal{T}(H_\infty)$ fixes each point in the image of $\mathcal{T}(\Delta/G_n)$ in $\mathcal{T}(H_\infty)$, where $A \in G_n - G_{n+1}$.*

*Proof.* Let $\mu$ be the lift to $H_\infty$ of a Beltrami coefficient on the Riemann surface $\Delta/G_n$. Then there exists a lift $\tilde{\mu}$ of $\mu$ to $\Delta \times \widehat{G}$ such that

$$\tilde{\mu}(z,t) = \tilde{\mu}(z,Bt)$$

for all $B \in G_n$ and for all $t \in \widehat{G}$. Since $A \in G_n$, it follows that

$$\tilde{\hat{A}}^{*}\tilde{\mu}(z,t) = \tilde{\mu}(Az, AtA^{-1})\frac{\bar{A}'(z)}{A'(z)} = \tilde{\mu}(z,At) = \tilde{\mu}(z,t).$$

Thus $\hat{A}$ fixes the image of $\mathcal{T}(\Delta/G_n)$ in $\mathcal{T}(H_\infty)$. □

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## References

[1] L. Bers, *Universal Teichmüller Space*, Analytic methods in mathematical physics (Sympos., Indiana Univ., Bloomington, Ind., 1968), pp. 65–83. Gordon and Breach, New York, 1970. MR0349988 (50:2481)

[2] I. Biswas and S. Nag, *Weil-Petersson geometry and determinant bundles on inductive limits of moduli spaces*, Lipa's legacy (New York, 1995), pp. 51-80, Contemp. Math., vol. 211. MR1476981 (99m:14050)

[3] I. Biswas and S. Nag, *Limit constructions over Riemann surfaces and their parameter spaces, and the commensurability group action*, Sel. Math., new ser. 6 (2000), pp. 185-224. MR1816860 (2002f:32026)

[4] A. Candel, *Uniformization of surface laminations*, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 4, pp. 489-516. MR1235439 (94f:57025)

[5] A. Douady and C. J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986), no. 1-2, 23-48. MR0857678 (87j:30041)

[6] C. Earle and C. McMullen, *Quasiconformal isotopies*, Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986), 143-154, Math. Sci. Res. Inst. Publ., no. 10, Springer, New York, 1988. MR0955816 (89h:30028)

[7] É. Ghys, *Laminations par surfaces de Riemann*, Dynamique et géométrie complexes (Lyon, 1997), ix, xi, 49-95, Panor. Synthèses, 8, Soc. Math. France, Paris, 1999. MR1760843 (2001g:37068)

[8] S. Kerckhoff, *The Nielsen Realization Problem*, Ann. of Math. 117, 235-265, 1983. MR0690845 (85e:32029)

[9] G. A. Margulis, *Discrete groups of motions of manifolds of non-positive curvature*, Proc. Int. Cong. Math., vol. 2, Vancouver, 1974, pp. 21-34. MR0492072 (58:11226)

[10] S. Nag and D. Sullivan, *Teichmüller theory and the universal period mappings via quantum calculus and the $H^{1/2}$ space of the circle*, Osaka J. Math. 32 (1995), 1-34. MR1323099 (96c:32023)

[11] C. Odden, *Virtual automorphism group of the fundamental group of a closed surface*, Ph.D. Thesis, Duke University, Durham, 1997.

[12] D. Šarić, *On Quasiconformal Deformations of the Universal Hyperbolic Solenoid*, preprint, available at: http://math.usc.edu/$\sim$ saric.

[13] D. Sullivan, *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers*, Milnor Festschrift, Topological methods in modern mathematics (L. Goldberg and A. Phillips, eds.), Publish or Perish, 1993, 543-563. MR1215976 (94c:58060)

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